Math 210A Lecture 20 Notes

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1 Schreier's Refinement Theorem and Nilpotent Groups

1.1 Schreier's refinement theorem

Definition 1.1. A **refinement** of a subnormal series $(H_i)_{i=0}^t$ os a subnormal series $(K_j)_{j=0}^s$ usch that there exists an increasing function $f : \{0, \dots, t\} \to \{0, \dots, s\}$ with $H_i = K_{f(i)}$ for all i.

Definition 1.2. Two subnormal series $(H_i)_{i=0}^t$ and $(K_j)_{j=0}^s$ are **equivalent** if $s = t$ and there exists a permutation $\sigma \in S_t$ such that $H_i/H_{i-1} \cong K_{\sigma(i)}/K_{\sigma(i)-1}$ for all $i \in \{1, \dots, t\}$

Theorem 1.1 (Schreier refinement theorem)**.** *Any two subnormal series in a group G have equivalent refinements.*

Proof. Here is the idea of the proof. If $(H_i)_{i=0}^t$ and $(K_j)_{j=0}^s$ are subnormal series, let $N_{si+j} = H_i(H_{i+1} \cap K_j)$ for all $0 \le i < t$ and $0 \le j < s$ and $N_{st} = G$. This refines (H_i). Do the same for (K_j). To see that they are equivalent, use the butterfly (or Zassenhaus) lemma from homework. $\square$

1.2 Nilpotent groups

Definition 1.3. The **lower central series** of a group G is $G = G$. $G_{i+1} = [G, G_i]$, where $[G, G_i]$ is the subgroup generated by commutators, $\langle \{[a, b] : a \in G, b \in G_i\} \rangle$.

Definition 1.4. A group G is **nilpotent** if $G_n = 1$ for all sufficiently large n in the lower central series. The smallest n such that $G_{n+1} = 1$ is the **nilpotence class** of G

Example 1.1. Let $E_{i,j}(\alpha)$ be the elementary matrix $I + \alpha e_{i,j}$.

1. $E_{i,j}(\alpha)E_{i,j}(\beta) = E_{i,j}(\alpha + \beta)$.
2. If $i \neq j$, $k \neq \ell$, and $i \neq \ell$, then

$$[E_{i,j}(\alpha), E_{k,\ell}(\beta)] = \begin{cases} E_{i,\ell}(\alpha\beta) & j = k \\ 0 & j \neq k. \end{cases}$$

3. Let U be the group of upper triangular matrices with 1s along the diagonal. Then $U = \langle \{E_{i,j}(\alpha) : i < j, \alpha \in F\} \rangle$. $U_2 = U'$ is the subgroup of such matrices with 0s on the diagonal above the main diagonal. U_3 is the subgroup of such matrices with 0s on the 2 diagonals above the main diagonal. Continuing like this, we get $U_n = 1$.

Example 1.2. Let

$$G = \mathrm{Aff}(F) = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a \in F^*, b \in F \right\} \cong F \rtimes F^*,$$

where the subgroups in the direct product are the off-diagonal matrices (with 1s in the diagonal) and the subgroup of diagonal matrices.

$$\left[\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right] = \begin{bmatrix} 1 & ab \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b(a-1) \\ 0 & 1 \end{bmatrix},$$

so

$$U = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} = [G, G]$$

if $F \not\cong F_2$. $G'' = 1$, and $G_n = U$ for all $n \geq 2$. So G is solvable but not nilpotent.

Definition 1.5. The **upper central series** $(Z^i(G))_{i \geq 0}$ of a group G is $Z^0(G) = 1$, $Z^i(G) = Z(G)$, and $G^{i+1}(G)$ is the inverse simage of $Z(G/Z^i(G))$ under the quotient map $G \to G/Z^i(G)$.

Proposition 1.1. *G is nilponent if and only if the upper central series is finite. If n is minimal such that $G_{n+1} = 1$, then $G_{n+1-i} \leq Z^i(G)$ for all i, and $Z^n(G)$ is minimal such that $Z^n(G) = G$.*

Proof. This is proven by induction. Here is the idea. Let $G = G_1 > G_2 > \cdots > G_n > G_{n+1} = 1$. Then $[G, G_n] = 1$, so $G_n \leq Z(G) = Z_1(G)$. $\square$

Example 1.3. Nilpotent groups can have different upper and lower central series. Look at $G = \mathbb{Z}/p\mathbb{Z} \times U$, where U is the set of upper triangular 4×4 matrices with 1s on the diagonal and entries in $\mathbb{F}_p$. THen $G_2 = U_2$, $G_3 = U_3$¡ and $G_4 = 1$. $Z^1(G) = Z(G) = \mathbb{Z}/p\mathbb{Z} \times U_3$, $Z^2(G) = \mathbb{Z}/p\mathbb{Z} \times U_2$, and $Z^3(G) = \mathbb{Z}/p\mathbb{Z} \times U_1 = G$.

Proposition 1.2. *Finite p-groups are nilpotent.*

Proof. Let P be a finite p-group. We induct on $|P| \neq 1$. Then $Z(P) \neq 1$, so $P/Z(P)$ is a p-group o smaller order so it is nilpotent. Say $\overline{P} = P/Z(P)$ has niltpotence class n. THen $Z^n(P/Z(P)) = P/Z(P) = \overline{P}$. Let $|pi_i : P \to P/Z^i(P)$. THen $Z^{i+1}(P) = \pi_i^{-1}(Z(P/Z^i(P))) = \pi_i^1(Z(\overline{P}/(Z^i(P)/Z(P))))$. By induction, $Z^i(P)/Z(P) = Z^{i-1}(\overline{P})$, so this is equal to $\pi_1^{-1}(Z^{i+1}(P))$. So the smallest j such that $Z^j(P) = P$ is $j = n+1$. $\square$